TENTAMEN IMAGE PROCESSING

7-04-2008

EXAMPLE TEST

Put your name on all pages which you hand in, and number them. Write the total number of pages you hand in on the first page. Write clearly and not with pencil or red pen. You can answer in English or Dutch. ***Always motivate your answers.*** *You get 10 points for free. Success!*

Problem 1 (25 pt)
Discrete Laplacian filtering of two variables is given by

$$g(x, y) = f(x + 1, y) + f(x - 1, y) + f(x, y + 1) + f(x, y - 1) - 4f(x, y)$$

where $f(x, y)$ is the input and $g(x, y)$ the output image.

a. Show that Laplacian filtering is a linear operation.

b. Give the 2-D filter mask $h(x, y)$ corresponding to this operation.

c. Give the equivalent filter $H(u, v)$ that implements this operation in the frequency domain. Assume that the input image has size $M \times N$.

d. The frequency domain filter satisfies $H(0, 0) = 0$ (check that your answer in **c**. satisfies this). What property of the Laplace filter in the spatial domain does this formula correspond to?

e. Is the Laplacian filter a low-pass or high-pass filter? Explain in terms of the behaviour of $H(u, v)$.

f. Laplacian filtering is very sensitive to noise. Explain why and give a possible remedy.

Problem 2 (25 pt)
Consider a binary image X with 4-connected 1-pixels and 8-connected 0-pixels.

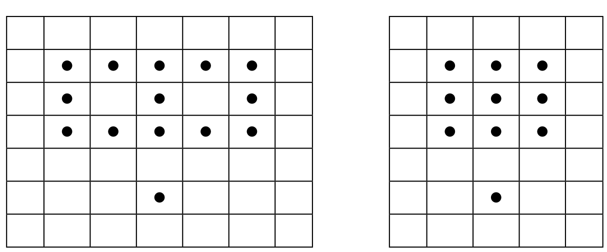
a. We want to select isolated 1-pixels (1-pixels without 4-connected 1-pixels as neighbour) by a hit-or-miss transformation

$$\psi(X) := X \circledast (A^1, A^2).$$

Give a structuring element pair (A^1, A^2) which achieves this selection.

b. How does the number of 1-components (connected components of 1-pixels) change under this hit-or-miss transformation? Same question for the genus (Euler number) g_4, which is the number of 1-components minus the number of holes (connected components of 0-pixels).

Check your answers by the images in the figure below.



(a) (b)

Figure 1: *Binary images with isolated 1-pixels.*

Give for both images:

1. the number of 1-components of X and $\psi(X)$;
2. the genus $g_4(X)$ and $g_4(\psi(X))$.

c. Is the transformation ψ you have found an increasing mapping? If not, give a counterexample.

Problem 3 (20 pt)
Consider the simple 2-bit image:

$$\begin{array}{cccc} 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \end{array}$$

a. Compute the entropy of this image.

b. Now consider encoding pairs of pixels which are horizontal neighbours instead of single pixels. Assume that the last pixel of a row is connected to the first pixel in that row, so that there are 16 horizontal neighbouring pixel pairs. Again compute the entropy, now per pixel pair.

c. Divide the result in **b.** by 2 to get the entropy per pixel. Why is this entropy smaller than found in **a.**?

Problem 4 (20 pt)
A simple global iterative threshold selection algorithm is defined by the following steps. Here k is an integer denoting the iteration number.

1. Put $k = 0$. Select an initial estimate for the global threshold $T(0)$.
2. Increase k by 1. In iteration k:
   - **a** segment the image using the global threshold $T(k-1)$. This produces two groups of pixels, G_1 and G_2, consisting of all pixels with values $> T(k-1)$ and $\leq T(k-1)$, respectively.
   - **b** compute the mean intensity values $m_1(k)$ and $m_2(k)$ for the pixels in G_1 and G_2, respectively.
   - **c** compute a new threshold value:

$$T(k) = \frac{m_1(k) + m_2(k)}{2}$$

3. *Repeat* step 2 until the change $|T(k) - T(k-1)|$ is smaller than a predefined parameter value ΔT.

a. Restate this algorithm so that it uses the histogram of the image instead of the image itself.

b. The initial threshold should be chosen between the minimum and maximum values in the image. To see why, consider an image with a bimodal histogram whose intensities are all above $L/2$. Analyse what happens when the initial estimate is chosen as $T(0) = 0$.

Formula sheet

Co-occurrence matrix $g(i,j) = \{$no. of pixel pairs with grey levels (z_i, z_j) satisfying predicate $Q\}$, $1 \leq i,j \leq L$

Convolution, 2-D discrete $(f \star h)(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)\, h(x-m, y-n)$, for $x = 0, 1, 2, \ldots, M-1$, $y = 0, 1, 2, \ldots, N-1$

Convolution Theorem, 2-D discrete $\mathcal{F}\{f \star h\}(u,v) = F(u,v)\, H(u,v)$

Distance measures Euclidean: $D_e(p,q) = \sqrt{(p_1-q_1)^2 + (p_2-q_2)^2}$, City-block: $D_4(p,q) = |p_1 - q_1| + |p_2 - q_2|$, Chessboard: $D_8(p,q) = \max(|p_1 - q_1|, |p_2 - q_2|)$

Entropy, source $H = -\sum_{j=1}^{J} P(a_j) \log P(a_j)$

Entropy, estimated for L-level image: $\tilde{H} = -\sum_{k=0}^{L-1} p_r(r_k) \log_2 p_r(r_k)$

Error, root-mean square $e_{\text{rms}} = \left[\frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left(\widehat{f}(x,y) - f(x,y) \right)^2 \right]^{\frac{1}{2}}$

Exponentials $e^{ix} = \cos x + i \sin x$; $\quad \cos x = (e^{ix} + e^{-ix})/2$; $\quad \sin x = (e^{ix} - e^{-ix})/2i$

Filter, inverse $\widehat{\mathbf{f}} = \mathbf{f} + \mathbf{H}^{-1}\mathbf{n}$, $\widehat{F}(u,v) = F(u,v) + \frac{N(u,v)}{H(u,v)}$

Filter, parametric Wiener $\widehat{\mathbf{f}} = \left(\mathbf{H}^t\mathbf{H} + K\,\mathbf{I}\right)^{-1} \mathbf{H}^t\mathbf{g}$, $\widehat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + K} \right] G(u,v)$

Fourier series of signal with period T: $f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{i \frac{2\pi n}{T} t}$, with Fourier coefficients: $c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, e^{-i \frac{2\pi n}{T} t}\, dt, \quad n = 0, \pm 1, \pm 2, \ldots$

Fourier transform 1-D (continuous) $F(\mu) = \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi\mu t}\, dt$

Fourier transform 1-D, inverse (continuous) $f(t) = \int_{-\infty}^{\infty} F(\mu)\, e^{i2\pi\mu t}\, d\mu$

Fourier Transform, 2-D Discrete $F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-i2\pi(u\,x/M + v\,y/N)}$ for $u = 0, 1, 2, \ldots, M-1$, $v = 0, 1, 2, \ldots, N-1$

Fourier Transform, 2-D Inverse Discrete $f(x,y) = \frac{1}{M\,N} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v)\, e^{i2\pi(u\,x/M + v\,y/N)}$ for $x = 0, 1, 2, \ldots, M-1, y = 0, 1, \ldots, N-1$

Fourier spectrum Fourier transform of $f(x,y)$: $F(u,v) = R(u,v) + i\,I(u,v)$, Fourier spectrum: $|F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)}$, phase angle: $\phi(u,v) = \arctan\left(\frac{I(u,v)}{R(u,v)}\right)$

Gaussian function mean μ, variance σ^2: $G_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2}$

Gradient $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

Histogram $h(m) = \#\{(x,y) \in D : f(x,y) = m\}$. Cumulative histogram: $\mathsf{P}(\ell) = \sum_{m=0}^{\ell} h(m)$

Impulse, discrete $\delta(0) = 1, \delta(x) = 0$ for $x \in \mathbb{N} \backslash \{0\}$

Impulse, continuous $\delta(0) = \infty, \delta(x) = 0$ for $x \neq 0$, with $\int_{-\infty}^{\infty} f(t)\, \delta(t - t_0)\, dt = f(t_0)$

Impulse train $s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$, with Fourier transform $S(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta(\mu - \frac{n}{\Delta T})$

Laplacian $\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

Laplacian-of-Gaussian $\nabla^2 G_\sigma(x,y) = -\frac{2}{\pi\sigma^4} \left(1 - \frac{r^2}{2\sigma^2}\right) e^{-r^2/2\sigma^2} \quad (r^2 = x^2 + y^2)$

Morphology

Dilation $\delta_A(X) = X \oplus A = \bigcup_{a\in A} X_a = \bigcup_{x\in X} A_x = \{h \in E : \check{A}_h \cap X \neq \emptyset\}$,
where $X_h = \{x + h : x \in X\}$, $h \in E$ and $\check{A} = \{-a : a \in A\}$

Erosion $\varepsilon_A(X) = X \ominus A = \bigcap_{a\in A} X_{-a} = \{h \in E : A_h \subseteq X\}$

Opening $\gamma_A(X) = X \circ A := (X \ominus A) \oplus A = \delta_A\varepsilon_A(X)$

Closing $\phi_A(X) = X \bullet A := (X \oplus A) \ominus A = \varepsilon_A\delta_A(X)$

Hit-or-miss transform $X \circledast (B_1, B_2) = (X \ominus B_1) \cap (X^c \ominus B_2)$

Thinning $X \otimes B = X \backslash (X \circledast B)$, **Thickening** $X \odot B = X \cup (X \circledast B)$

Morphological reconstruction Marker F, mask G, structuring element B:
$X_0 = F$, $X_k = (X_{k-1} \oplus B) \cap G, \quad k = 1, 2, 3, \ldots$

Morphological skeleton Image X, structuring element B: $SK(X) = \bigcup_{n=0}^{N} S_n(X)$,
$S_n(X) = X \underset{n}{\ominus} B \setminus (X \underset{n}{\ominus} B) \circ B$, where $X \underset{0}{\ominus} B = X$ and N is the largest integer such that $S_N(X) \neq \emptyset$

Grey value dilation $(f \oplus b)(x, y) = \max\limits_{(s,t)\in B} [f(x - s, y - t) + b(s, t)]$

Grey value erosion $(f \ominus b)(x, y) = \min\limits_{(s,t)\in B} [f(x + s, y + t) - b(s, t)]$

Grey value opening $f \circ b = (f \ominus b) \oplus b$

Grey value closing $f \bullet b = (f \oplus b) \ominus b$

Morphological gradient $g = (f \oplus b) - (f \ominus b)$

Top-hat filter $T_{\text{hat}} = f - (f \circ b)$, **Bottom-hat filter** $B_{\text{hat}} = (f \bullet b) - f$

Sampling of continuous function $f(t)$: $\tilde{f}(t) = f(t)\, s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\, \delta(t - n\Delta T)$.
Fourier transform of sampled function: $\tilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F(\mu - \frac{n}{\Delta T})$

Sampling theorem Signal $f(t)$, bandwidth $\mu_{\max}$: If $\frac{1}{\Delta T} \geq 2\mu_{\max}$, $f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta T)\, \mathrm{sinc}\left[\frac{t-n\Delta T}{n\Delta T}\right]$.

Sampling: downsampling by a factor of 2: $\downarrow_2 (a_0, a_1, a_2, \ldots, a_{2N-1}) = (a_0, a_2, a_4, \ldots, a_{2N-2})$

Sampling: upsampling by a factor of 2: $\uparrow_2 (a_0, a_1, a_2, \ldots, a_{N-1}) = (a_0, 0, a_1, 0, a_2, 0, \ldots, a_{N-1}, 0)$

Set, circularity ratio $R_c = \frac{4\pi A}{P^2}$ of set with area A, perimeter P

Set, diameter $\mathrm{Diam}(B) = \max\limits_{i,j} [D(p_i, p_j)]$ with p_i, p_j on the boundary B and D a distance measure

Sinc function $\mathrm{sinc}\,(x) = \frac{\sin(\pi x)}{\pi x}$ when $x \neq 0$, and $\mathrm{sinc}\,(0) = 1$. If $f(t) = A$ for $-W/2 \leq t \leq W/2$ and zero elswhere (block signal), then its Fourier transform is $F(\mu) = A\, W \mathrm{sinc}\,(\mu\, W)$

Spatial moments of an $M \times N$ image $f(x, y)$: $m_{pq} = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} x^p\, y^q\, f(x, y), \quad p, q = 0, 1, 2, \ldots$

Statistical moments of distribution $p(i)$: $\mu_n = \sum_{i=0}^{L-1} (i - m)^n\, p(i)$, $m = \sum_{i=0}^{L-1} i\, p(i)$

Signal-to-noise ratio, mean-square $\mathrm{SNR}_{\mathrm{rms}} = \frac{\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \hat{f}(x,y)^2}{\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left(\hat{f}(x,y) - f(x,y)\right)^2}$

Wavelet decomposition with low pass filter h_ϕ, band pass filter h_ψ. For $j = 1, \ldots, J$:
Approximation: $c_j = \mathbf{H}c_{j-1} = \downarrow_2 (h_\phi * c_{j-1})$; Detail: $d_j = \mathbf{G}c_{j-1} = \downarrow_2 (h_\psi * c_{j-1})$

Wavelet reconstruction with low pass filter $\tilde{h}_\phi$, band pass filter $\tilde{h}_\psi$. For $j = J, J - 1, \ldots, 1$:
$c_{j-1} = \tilde{h}_\phi * (\uparrow_2 c_j) + \tilde{h}_\psi * (\uparrow_2 d_j)$

Wavelet, Haar basis $h_\phi = \frac{1}{\sqrt{2}}(1, 1)$, $h_\psi = \frac{1}{\sqrt{2}}(1, -1)$, $\tilde{h}_\phi = \frac{1}{\sqrt{2}}(1, 1)$, $\tilde{h}_\psi = \frac{1}{\sqrt{2}}(1, -1)$

Answers

Problem 1

a. Let the input be a linear combination of two input images: $f(x,y) = a\, f_1(x,y) + b\, f_2(x,y)$. Then

$$\begin{aligned} g(x,y) &= a\, f_1(x+1,y) + b\, f_2(x+1,y) + a\, f_1(x-1,y) + b\, f_2(x-1,y) + a\, f_1(x,y+1) \\ &\quad + b\, f_2(x,y+1) + a\, f_1(x,y-1) + b\, f_2(x,y-1) - 4a\, f_1(x,y) - 4b\, f_2(x,y) \\ &= a\left[f_1(x+1,y) + f_1(x-1,y) + f_1(x,y+1) + f_1(x,y-1) - 4f_1(x,y)\right] \\ &\quad + b\left[f_2(x+1,y) + f_2(x-1,y) + f_2(x,y+1) + f_2(x,y-1) - 4f_2(x,y)\right] \\ &= a\, g_1(x,y) + b\, g_2(x,y) \end{aligned}$$

So the output is the same linear combination of the outputs of the individual input images. Hence the operation is linear.

b. The mask is:

0	1	0
1	-4	1
0	1	0

c. The frequency domain representation $H(u,v)$ is the DFT of the spatial filter kernel $h(x,y)$. If the center of the mask is assumed to be at $(0,0)$, then we see that $h(0,0) = -4$, $h(1,0) = h(-1,0) = h(0,1) = h(0,-1) = 1$. So, working with positive and negative indices (compare Fig. 4.23 of the course book), we get

$$\begin{aligned} H(u,v) &= \sum_{x=-M/2}^{M/2-1} \sum_{y=-N/2}^{N/2-1} h(x,y)\, e^{-i2\pi(u\,x/M + v\,y/N)} \\ &= -4 + e^{i2\pi u/M} + e^{-i2\pi u/M} + e^{i2\pi v/N} + e^{-i2\pi v/N} \\ &= -4 + 2\cos(2\pi u/M) + 2\cos(2\pi v/N) \end{aligned}$$

The centered version of the filter transfer function is:

$$H(u,v) = -4 + 2\cos(2\pi[u - M/2]/M) + 2\cos(2\pi[v - N/2]/N)$$

Now the center is at $(M/2, N/2)$.

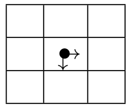
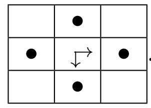
This form can also be obtained by working with nonnegative indices only. In that case $h(x,y)$ is first multiplied by $(-1)^{x+y}$ before computing the DFT (compare section 4.7.3 of the course book).

d. $H(0,0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} h(x,y)$, i.e., $H(0,0) = 0$ means that the sum of the coefficients of the Laplace filter in the spatial domain is zero. Since $G(u,v) = H(u,v)F(u,v)$ it also means that the sum of the pixel values of the filtered image $g(x,y)$ will be zero.

e. Laplacian filtering emphasizes sharp transitions in images, so it is a high-pass filter. This is reflected in the (un-centered) transfer function: $H(u,v) = 0$ at the origin and its magnitude increases when $|u|$ or $|v|$ increase. So the DC-component is suppressed and higher frequencies are passed, which is the characteristic of a high-pass filter.

f. Laplacian filtering is a discrete version of a second order derivative. So it will also emphasize noise pixels, which represent local transitions in grey value. A possible remedy is to low-pass filter the image before taking the derivative, for example by Gaussian smoothing. (Equivalently, applying the Laplacian of a Gaussian function.)

Problem 2

a. Take $A^1 =$  $A^2 =$ .

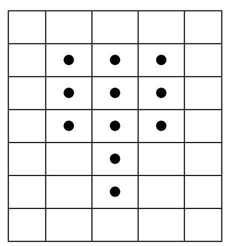
b. The number of 1-components decreases, genus can increase or decrease.
Fig. 1(a):

1. number of 1-components of X: 2; of $\psi(X)$: 1;
2. genus $g_4(X) = 2 - 2 = 0$; and $g_4(\psi(X)) = 1 - 0 = 1$.

Fig. 1(b):

1. number of 1-components of X: 2; of $\psi(X)$: 1;
2. genus $g_4(X) = 2 - 0 = 2$; and$g_4(\psi(X)) = 1 - 0 = 1$.

c. ψ is not increasing. Counterexample: create an image Y by adding one 1-pixel to Fig. 1(b) (picture below). Now $\psi(Y) = \emptyset$.



Problem 3

a. We can make the following table:

Intensity	Count	Probability
1	8	$\frac{1}{2}$
2	4	$\frac{1}{4}$
3	4	$\frac{1}{4}$

The entropy of the image is thus:

$$\begin{aligned}
\tilde{H} &= -\sum_{k=1}^{L-1} p_r(r_k) \log_2 p_r(r_k) \\
&= -\Big[\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{4}\log_2\frac{1}{4} + \frac{1}{4}\log_2\frac{1}{4}\Big] \\
&= -\Big[\frac{1}{2}\cdot(-1) + \frac{1}{4}\cdot(-2) + \frac{1}{4}\cdot(-2)\Big] = \frac{3}{2} \text{ bit/pixel}
\end{aligned}$$

b. The Now we can make the following table:

Intensity pair	Count	Probability
(1,1)	4	$\frac{1}{4}$
(1,2)	4	$\frac{1}{4}$
(2,3)	4	$\frac{1}{4}$
(3,1)	4	$\frac{1}{4}$

The entropy is thus:

$$\begin{aligned}\tilde{H} &= -\sum_{k=1}^{L-1} p_r(r_k)\log_2\, p_r(r_k)\\ &= -4\Big[\frac{1}{4}\log_2\frac{1}{4}\Big] = 2 \text{ bit/pixel pair}\end{aligned}$$

c. The entropy per pixel is thus 1 bit/pixel. It is smaller than found in **a.** because the intensity values are not statistically independent, but are correlated.

Problem 4

a. Let $p_i = n_i/n$, or $0 \leq i \leq L-1$, where n_i is the number of pixels with intensity i, n is the total number of pixels in the image, and L the number of intensities. In step k the means can be computed by

$$m_1(k) = \frac{1}{P_1(k)}\sum_{i=0}^{I(k-1)} ip_i, \quad m_2(k) = \frac{1}{P_2(k)}\sum_{i=I(k-1)+1}^{L-1} ip_i$$

where

$$P_1(k) = \sum_{i=0}^{I(k-1)} p_i, \quad P_2(k) = \sum_{i=I(k-1)+1}^{L-1} p_i$$

and $I(k-1)$ is the smallest integer less than or equal to $T(k-1)$.

b. Let $T(0) = 0$. Since all image values are greater than $L/2$, all pixels will be assigned to group G_1. So $m_1(1)$ will be the mean value, say M, of the image and $m_2(1)$ will be 0. Hence $T(1)$ will be $M/2$. But $M < L$, so $M/2 < L/2$. This means that there will be no pixels with values smaller than $T(1)$. Hence $m_1(2) = M$, $m_2(2) = 0$ and again the threshold $T(2) = M/2$. So the algorithm will terminate with the (wrong) threshold $M/2$.